TORSION IN TENSOR PRODUCTS, AND TENSOR POWERS, OF MODULES

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ABSTRACT. For finitely generated modules M and N over a complete intersection R, the vanishing of $\operatorname{Tor}_i^R(M,N)$ for all $i \geq 1$ gives a tight relationship among depth properties of M, N and $M \otimes_R N$. Here we concentrate on the converse and show, under mild conditions, that $M \otimes_R N$ being torsion-free (or satisfying higher Serre conditions) forces vanishing of Tor. Special attention is paid to the case of tensor powers of a single module.

1. INTRODUCTION

In his 1961 seminal paper "Modules over unramified regular local rings" [2], Auslander studied torsion in tensor products of nonzero finitely generated modules M and N over unramified regular local rings R. Under the assumption that $M \otimes_R N$ is torsion-free, he proved:

(1) M and N must be torsion-free, and
(2) M and N are *Tor-independent*, that is, $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$.

In the same paper he observed the following formula

$$\operatorname{pd}_R(M \otimes_R N) = \operatorname{pd}_R(M) + \operatorname{pd}_R(N)\,,$$

for Tor-independent modules over *arbitrary* local rings, as long as both M and N have finite projective dimension. In view of the Auslander-Buchsbaum Formula [3, Theorem 3.7], this is equivalent to the *depth formula*:

$$\operatorname{depth}_R(M) + \operatorname{depth}_R(N) = \operatorname{depth}(R) + \operatorname{depth}_R(M \otimes_R N) \tag{1.0.1}$$

The depth formula is true for Tor-independent modules over complete intersections [21, Proposition 2.5] and, more generally, over arbitrary local rings if one of the modules has finite complete intersection dimension; see [1], [24]. We know of no counterexamples for Tor-independent modules over arbitrary local rings.

In this paper we continue the theme of Auslander's paper, obtaining consequences of the assumption that $M \otimes_R N$, or perhaps $\otimes_R^n M$, is torsion-free (or satisfies certain Serre conditions), but over more general local rings. Section 2 establishes terminology and discusses tools of the trade. Section 3 shows, at least over a domain, that high tensor powers of non-free modules must have torsion. A point of interest is that the torsion element we identify comes from a universal example.

Date: February 8, 2013.
Key words and phrases. complete intersection, η pairing, Frobenius endomorphism, tensor product, torsion.
SBI partly supported by NSF grant DMS-1201889 and a Simons Fellowship; RW partly supported by a Simons Collaboration Grant.

In §4–§7 we concentrate on complete intersections. The main result of §4 is Theorem 4.7, which says that vanishing of $\operatorname{Tor}_1^R(M,N)$, together with certain Serre conditions on the modules and their tensor product, forces the vanishing of $\operatorname{Tor}_i^R(M,N)$ for all $i \geq 1$. In §5 we deploy an intersection pairing η developed by H. Dao [13]. Combining the vanishing of the η-pairing with certain Serre conditions, we establish the vanishing of $\operatorname{Tor}_i^R(M,N)$ for all $i \geq 1$. Given the vanishing of η, we are able to sharpen the existing results in the literature, roughly speaking, "by one", e.g., by weakening the hypothesis (S_{n+1}) to (S_n). For an even-dimensional graded hypersurface with an isolated singularity, the vanishing of η is known [31]. It is conjectured in [32] that the η-pairing *always* vanishes for isolated singularities that are complete intersections of codimension $c \geq 2$. In §6 we find situations where η is known to vanish, and so obtain new results on vanishing of Tor.

In Section 7 we consider the Frobenius endomorphism $\varphi : R \to R$ over F-finite rings of characteristic p. We show that if R is not regular then $R^{\varphi} \otimes_R {}^{\varphi}M$ has torsion for *every* nonzero finitely generated R-module M.

2. Depth, torsion, and Serre's conditions

In this section we recall basic concepts and results, mainly involving depth, that are used in this article. For details, see [6, Chapter 1]. We assume throughout the paper that R is a commutative Noetherian ring.

2.1. Depth. For an ideal I of R, the *I-depth* of M is the number

$$\operatorname{depth}_R(I,M) = \inf\{n \geq 0 \mid \operatorname{Ext}_R^n(R/I,M) \neq 0\}\,. \tag{2.1.1}$$

Thus $\operatorname{depth}_R(I,M)$ is finite if and only if $\operatorname{Ext}_R^*(R/I,M) \neq 0$, and $\operatorname{depth}_R(I,M) = 0$ if and only if $\operatorname{Hom}_R(R/I,M) \neq 0$; equivalently, $\Gamma_I(M) \neq 0$ where

$$\Gamma_I(M) = \{m \in M \mid I^s M = 0 \text{ for some } s \geq 0\}$$

is the I-torsion submodule of M. The I-depth of M can also be computed from its local cohomology modules, $\mathrm{H}_I^n(M)$, with respect to I:

$$\operatorname{depth}_R(I,M) = \inf\{n \geq 0 \mid \mathrm{H}_I^n(M) \neq 0\}\,.$$

If $\underline{x} := x_1, \dots, x_d$ is a sequence of elements in R, and K is the Koszul complex on $\underline{x}$, then the $(\underline{x})$-depth of M may be computed from its Koszul homology:

$$\operatorname{depth}_R((\underline{x}),M) = d - \sup\{i \geq 0 \mid \mathrm{H}_i(K \otimes_R M) \neq 0\}\,.$$

This is the *depth sensitivity* of the Koszul complex.

If R is local (commutative and Noetherian) with maximal ideal $\mathfrak{m}$, we write $\operatorname{depth}_R M$ for the $\mathfrak{m}$-depth of M and call it the *depth* of M; the depth of R is, not surprisingly, denoted $\operatorname{depth} R$.

2.2. Regular sequences. When $\underline{x}$ is a regular sequence in R, one can compute the $(\underline{x})$-depth of M from a projective resolution, say F, of M as follows:

$$\operatorname{depth}_R((\underline{x}),M) = d - \sup\{i \geq 0 \mid \operatorname{Tor}_i^R(R/(\underline{x}),M) \neq 0\}\,. \tag{2.2.1}$$

Indeed, when $\underline{x}$ is regular, the associated Koszul complex K is a free resolution of $R/(\underline{x})$, and there are quasi-isomorphisms

$$(R/(\underline{x})) \otimes_R F \xleftarrow{\simeq} K \otimes_R F \xrightarrow{\simeq} K \otimes_R M$$

Thus the desired equality follows from the depth sensitivity of the Koszul complex. Note that $\mathrm{H}_*((R/(\underline{x})) \otimes_R F)$ is none other than $\operatorname{Tor}_*^R(R/(\underline{x}),M)$.

2.3. Torsion submodule. The *torsion submodule* $\top_R M$ of M is the kernel of the natural homomorphism $M \to \mathrm{Q}(R) \otimes_R M$, where $\mathrm{Q}(R)$ is the total quotient ring of R. The inclusion $\top_R M \subseteq M$ gives rise to an exact sequence

$$0 \longrightarrow \top_R M \longrightarrow M \longrightarrow \perp_R M \longrightarrow 0\,. \tag{2.3.1}$$

We say that M is *torsion* if $\top_R M = M$ (that is, $M_\mathfrak{p} = 0$ for each $\mathfrak{p} \in \mathrm{Ass}(R)$), and that it is *torsion-free* if $\top_R M = 0$.

Notation 2.4. For any non-negative integer n, set

$$\mathbb{X}^n(R) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{height} \mathfrak{p} \le n\}.$$

Some authors use this notation for the set of primes for which $\operatorname{depth}(R_\mathfrak{p}) \le n$; for Cohen-Macaulay rings, there is no conflict.

In what follows, the notation $(R, \mathfrak{m})$ will mean that R is a local ring, with maximal ideal $\mathfrak{m}$. The space $\mathbb{X}^{d-1}(R) = \operatorname{Spec} R \backslash \{\mathfrak{m}\}$, with d the (Krull) dimension of R, is often called the *punctured spectrum* of R.

We say that M is *locally free* on a subset U of $\operatorname{Spec} R$ if the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is free for each $\mathfrak{p} \in U$. It is easy to test torsion-freeness of modules locally free on the punctured spectrum:

Lemma 2.5. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of positive dimension and M a finitely generated R-module that is locally free on the punctured spectrum. Then M is torsion-free if and only if $\Gamma_\mathfrak{m}(M) = 0$, if and only if $\operatorname{depth}_R M \ge 1$.* □

Let M be a finitely generated R-module; M^* denotes it dual $\operatorname{Hom}_R(M, R)$. The module M is *torsionless* if it embeds in a free module, equivalently, the canonical map $M \to M^{**}$ is injective. Torsionless modules are torsion-free, and the converse holds if $R_\mathfrak{p}$ is Gorenstein for every associated prime $\mathfrak{p}$ of R; see [35, Theorem A.1]. The module M is *reflexive* provided the map $M \to M^{**}$ is an isomorphism.

2.6. Serre's conditions. Fix an integer $n \ge 0$. We say that M satisfies *Serre's condition* (S_n) if

$$\operatorname{depth}(M_\mathfrak{p}) \ge \min\{n, \dim R_\mathfrak{p}\} \quad \text{for each } \mathfrak{p} \in \operatorname{Spec}(R).$$

If R is Gorenstein, then M is torsion-free, respectively reflexive, if and only if it satisfies Serre's condition (S_1), respectively (S_2); see [27, Appendix A, §1].

An R-module M is *maximal Cohen-Macaulay* (abbreviated to MCM) provided $\operatorname{depth} M = \dim R$; in particular, M is required to be nonzero. A nonzero R-module is MCM if and only if it satisfies (S_n) for some (equivalently, every) $n \ge \dim R$.

We recall a technique from [20, §1] for lowering the codimension of the ring, to facilitate inductive arguments. In what follows, $\nu_R(M)$ denotes the minimal number of generators of the R-module M.

2.7. Pushforward and quasi-lifting Let R be a Gorenstein local ring and M a finitely generated torsion-free R-module. Choose a surjection $\varepsilon^*\colon R^{(\nu)} \twoheadrightarrow M^*$ with $\nu = \nu_R(M^*)$. Applying $\operatorname{Hom}(-, R)$ to this surjection, we obtain an injection $\varepsilon\colon M^{**} \hookrightarrow R^{(\nu)}$. Let M_1 be the cokernel of the composition $M \hookrightarrow M^{**} \hookrightarrow R^{(\nu)}$. The exact sequence

$$0 \to M \to R^{(\nu)} \to M_1 \to 0 \tag{2.7.1}$$

(or any sequence obtained by this recipe) is called the *pushforward* of M. The extension (2.7.1) and the module M_1 are unique up to non-canonical isomorphism.

We sometimes refer to the module M_1 as the pushforward of M. Note that $M_1 = 0$ if and only if M is free.

Assume $R = S/(f)$ where $(S, \mathfrak{n})$ is a local ring and f is a non-zerodivisor in $\mathfrak{n}$. The *quasi-lifting* of M to S is the module E in the following exact sequence:

$$0 \to E \to S^{(\nu)} \to M_1 \to 0 \tag{2.7.2}$$

Here the map $S^{(\nu)} \twoheadrightarrow M_1$ is the composition of the canonical map $S^{(\nu)} \twoheadrightarrow R^{(\nu)}$ and the map $R^{(\nu)} \twoheadrightarrow M_1$ in (2.7.1). The quasi-lifting of M is unique up to isomorphism of S-modules.

We refer the reader to [20, §1] for basic properties of these constructions (cf. [9, Propositions 3.1 and 3.2]).

Proposition 2.8. *Let R be a Gorenstein local ring and M a finitely generated torsion-free R-module.*

(1) *Given any non-negative integer n, the module M satisfies Serre's condition (S_{n+1}) if and only if the pushforward M_1 satisfies (S_n).*
(2) *For $\mathfrak{p} \in \operatorname{Spec} R$, if the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is MCM, then $(M_1)_\mathfrak{p}$ is MCM or zero.*

Proof. When M_1 satisfies (S_n) the depth lemma [6, Proposition 1.2.9] implies that M satisfies (S_{n+1}). This proves one direction of (1); the converse is part of [20, Proposition 1.6]. The claim in (2) is a consequence of (1). $\square$

We say that M is an *n^{th} syzygy of N* provided there is an exact sequence

$$0 \to M \to F_{n-1} \to \cdots \to F_0 \to N \to 0\,,$$

in which each F_i is finitely generated and projective. If R is local and the ranks of the free modules F_i are chosen minimally, then M is uniquely determined by N up to isomorphism.

The following well-known characterization of n^{th} syzygy modules is a consequence of Proposition 2.8; see [27, Corollary A.12] for a more general statement.

Corollary 2.9. *Let R be a Gorenstein local ring and M a finitely generated R-module. Then M is an n^{th} syzygy if and only if M satisfies (S_n).*

Proof. Iterated use of the pushforward and Proposition 2.8 proves the "if" direction, and the converse follows from the depth lemma [6, Proposition 1.2.9]. $\square$

For some of our results we will need to know that our hypotheses, e.g., Serre's conditions (S_n), ascend along flat local homomorphisms. This can be problematic:

Example 2.10. The ring $\mathbb{C}[[x, y, u, v]]/(x^2, xy)$ has depth two and therefore, by Heitmann's theorem [17, Theorem 8], is the completion $\widehat{R}$ of a unique factorization domain $(R, \mathfrak{m})$. Then R, being normal, satisfies (S_2), but $\widehat{R}$ does not even satisfy (S_1), since the localization at the height-one prime ideal (x, y) has depth zero.

For Gorenstein rings, however, (S_n) *does* ascend and descend:

Lemma 2.11. *Let R and S be Gorenstein local rings and $R \to S$ a flat local homomorphism. Let M be a finitely generated R-module and $\mathfrak{p}$ a prime ideal of R.*

(1) *If M is torsion-free (respectively, reflexive), then so is the $R_\mathfrak{p}$-module $M_\mathfrak{p}$.*
(2) $(\top_R M)_\mathfrak{p} = \top_{R_\mathfrak{p}} M_\mathfrak{p}$.
(3) *M satisfies Serre's condition (S_n) as an R-module if and only if $S \otimes_R M$ satisfies (S_n) as an S-module.*

Proof. Part (1) is clear from the fact that M is torsion-free (respectively, reflexive) if and only if it is a first (respectively, second) syzygy, and (2) follows from (1).

Ascent in part (3) follows from the criterion of Corollary 2.9; and (S_1) descends since it is equivalent to the condition that $M \to M^{**}$ be injective. Now induction and Proposition 2.8 prove descent of (S_n). $\square$

3. Torsion in tensor powers

In this section we establish results on annihilators of elements in tensor powers of modules. Throughout, R will be a commutative ring.

Notation 3.1. Given elements $\underline{m} := m_1, \dots, m_d$ in an R-module M, we consider the element in $\otimes_R^d M$ defined by

$$\tau(\underline{m}) := \sum_{\sigma \in S_d} \operatorname{sign}(\sigma)\, m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(d)}\,,$$

where S_d denotes the permutations of the sequence $1, \dots, d$.

Proposition 3.2. *Let M be an R-module. If elements $m_1, \dots, m_d$ in M and $r_1, \dots, r_d$ in R satisfy*

$$r_1 m_1 + \cdots + r_d m_d = 0, \tag{3.2.1}$$

then $(r_1, \dots, r_d) \cdot \tau(\underline{m}) = 0$ in $\otimes_R^d M$.

Proof. The *twisted shuffle product* gives the graded R-algebra $\bigoplus_{n \geq 0} \otimes_R^n M$ a strictly skew-commutative structure; see [29, (12.4)]. Strictly skew-commutative means that for any $a \in \otimes_R^i M$ and $b \in \otimes_R^j M$, there are equalities

$$a \star b = (-1)^{ij} b \star a\,, \quad \text{and} \quad a \star a = 0 \text{ when } i \text{ is odd.}$$

By definition of the shuffle product, $\tau(\underline{m}) = m_1 \star \cdots \star m_d$. Thus for each j we have

$$\begin{aligned} r_j \cdot \tau(\underline{m}) &= m_1 \star \cdots \star m_{j-1} \star r_j m_j \star m_{j+1} \star \cdots \star m_n \\ &= -\sum_{i \neq j} r_i (m_1 \star \cdots \star m_{j-1} \star m_i \star m_{j+1} \star \cdots \star m_n) = -\sum_{i \neq j} r_i 0 = 0. \ \square \end{aligned}$$

There is a "universal" source for the element $\tau(\underline{m})$ in the following sense:

Remark 3.3. Consider the polynomial ring $\mathbb{Z}[\underline{x}]$ on indeterminates $\underline{x} := x_1, \dots, x_d$, and let U be the $\mathbb{Z}[\underline{x}]$-module with presentation

$$0 \longrightarrow \mathbb{Z}[\underline{x}] \xrightarrow{[x_1, \dots, x_d]^{\mathrm{t}}} \mathbb{Z}[\underline{x}]^d \longrightarrow U \longrightarrow 0$$

Let $u_1, \dots, u_d$ the the generators of U corresponding to the standard basis for $\mathbb{Z}[\underline{x}]^d$, so that $x_1 u_1 + \cdots + x_d u_d = 0$, i.e., $\underline{x}$ and $\underline{u}$ satisfy (3.2.1). Therefore $\operatorname{ann}_{\mathbb{Z}[\underline{x}]} \tau(\underline{u}) \supseteq (\underline{x})$, by Proposition 3.2; in fact, equality holds, by Theorem 3.6.

Given any R-module M with a syzygy relation (3.2.1), let $\varphi \colon \mathbb{Z}[\underline{x}] \to R$ be the homomorphism of rings with $x_i \mapsto r_i$, for each i, and extending the structure homomorphism $\mathbb{Z} \to R$. The hypothesis on M implies that there is a homomorphism of $\mathbb{Z}[\underline{x}]$-modules

$$f \colon U \longrightarrow M \quad \text{with } f(u_i) = m_i \text{ for } i = 1, \dots, d.$$

Under the induced map $\otimes^d f \colon \otimes_{\mathbb{Z}[\underline{x}]}^d U \to \otimes_R^d M$, the element $\tau(\underline{u})$ maps to $\tau(\underline{m})$.

The discussion below, culminating in Theorem 3.6, is prompted by this remark.

3.4. A Koszul syzygy module. Let R be a Noetherian ring, $\underline{r} := r_1, \dots, r_d$ a regular sequence in R with $(\underline{r}) \neq R$, and consider the complex

$$F := \quad 0 \longrightarrow R \xrightarrow{[r_1,\dots,r_d]^{\mathrm{t}}} R^d \longrightarrow 0$$

concentrated in degrees 0 and 1. Set $M = \mathrm{H}_0(F)$; as $\underline{r}$ is regular, F is a free resolution of M. Let $m_1, \dots, m_d$ the images in M of the standard basis for R^d.

The next result is in preparation for the proof of Theorem 3.6.

Lemma 3.5. *For each $n = 1, \dots, d$, the following statements hold:*

(1) *M and $\otimes_R^{n-1} M$ are* Tor*-independent.*
(2) *$\otimes_R^n F$ is a free resolution of $\otimes_R^n M$, and $\mathrm{pd}_R(\otimes_R^n M) = n$.*

Proof. The base case is $n = 1$, and then (1) and (2) are clear. Fix an integer n with $2 \leq n \leq d$, and assume these statements hold for all integers $\leq n-1$. Set $I = (\underline{r})$. For any positive integer s such that $\otimes_R^s F$ is a free resolution of $\otimes_R^s M$, one gets

$$\mathrm{Tor}_*^R(R/I, \otimes_R^s M) = \mathrm{H}_*((R/I) \otimes_R (\otimes_R^s F)) \cong ((\otimes_R^s (R/I) \otimes_R F))_*$$

where the last isomorphism holds because the complex in question has zero differential. In particular, $\mathrm{Tor}_s^R(R/I, \otimes_R^i M) \cong R/I \neq 0$, so that

$$\sup\{i \geq 0 \mid \mathrm{Tor}_i^R(R/(\underline{x}), \otimes_R^s M) \neq 0\} = s\,. \tag{3.5.1}$$

We can now complete the induction step.

(1) The induction hypothesis implies $\otimes_R^{n-1} F$ is a free resolution of $\otimes_R^{n-1} M$, so (3.5.1) and (2.2.1) give

$$\mathrm{depth}_R(I, \otimes_R^{n-1} M) = d - n - 1 \geq 1\,,$$

and also that $\mathrm{Tor}_*^R(M, \otimes_R^{n-1} M)$ is the homology of the complex

$$F \otimes_R (\otimes_R^{n-1} M) : \qquad 0 \longrightarrow \otimes_R^{n-1} M \xrightarrow{[\underline{r}]^{\mathrm{t}}} (\otimes_R^{n-1} M)^d \longrightarrow 0\,.$$

It follows that M and $\otimes_R^{n-1} M$ are Tor-independent.

(2) By hypothesis, F and $\otimes_R^{n-1} F$ are free resolutions of M and $\otimes_R^{n-1} M$, respectively. We have already proved, in (1), that these modules are Tor-independent, so the complex $F \otimes_R (\otimes_R^{n-1} F)$, that is to say, $\otimes_R^n F$, is a free resolution of $\otimes_R^n M$. In particular, one has $\mathrm{pd}_R(\otimes_R^n M) \leq n$; that equality holds follows from (3.5.1). $\square$

When R is a regular local ring and $\underline{r}$ generates its maximal ideal, part (1) of the next result is stated in [2, p. 638] and proved in [23, Proposition 3.1].

Theorem 3.6. *Let M and $\underline{r}$ be as in* 3.4*. The following statements hold:*

(1) *$\otimes_R^n M$ is torsion-free if and only if $n = 1, \dots, d-1$.*
(2) *The elements $\tau(\underline{m})$ in $\otimes_R^d M$ satisfies $\mathrm{ann}_R\, \tau(\underline{m}) = (\underline{r})$.*
(3) *The map $R/(\underline{r}) \to \otimes_R^d M$ of R-modules with $1 \mapsto \tau(\underline{m})$ induces a splitting*

$$\otimes_R^d M \cong R/(\underline{r}) \bigoplus W$$

where W is torsion-free; in particular, there are equalities

$$\Gamma_{(\underline{x})}(\otimes_R^d M) = \mathrm{Hom}_R(R/(\underline{x}), \otimes_R^d M) = R\tau(\underline{m}) \neq 0\,.$$

Proof. Set $I = (\underline{r})$ and fix a prime $\mathfrak{p} \in \operatorname{Spec} R$. If $I \not\subseteq \mathfrak{p}$, then the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is free, and hence so is $(\otimes_R^n M)_\mathfrak{p}$ for each $n \geq 1$. If $I \subseteq \mathfrak{p}$, then it follows from Lemma 3.5 that $(\otimes_R^n F)_\mathfrak{p}$ is a minimal free resolution of $(\otimes_R^n M)_\mathfrak{p}$, and hence that

$$\operatorname{depth}_{R_\mathfrak{p}}(\otimes_R^n M)_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} - n \geq d - n \quad \text{for } n = 1, \dots, d.$$

This proves that $\otimes_R^n M$ is torsion-free for $1 \leq n \leq d-1$. It will follow from (3) that $\otimes_R M^n$ has torsion for $n \geq d$.

As to parts (2) and (3), by construction $r_1 m_1 + \cdots + r_d m_d = 0$, so Proposition 3.2 gives an inclusion $I \subseteq \operatorname{ann}_R \tau(\underline{m})$. The reverse inclusion will follow, once we ascertain that the map in (3) splits. Consider the homomorphisms of R-modules

$$\otimes_R^d(F_0) \twoheadrightarrow \otimes_R^d M \twoheadrightarrow (\otimes_R^d M) \otimes_R R/I \cong \mathrm{H}_0((\otimes_R^d F) \otimes_R R/I) = \otimes_R^d(F_0 \otimes_R R/I)$$

where the surjections are the natural ones; the isomorphism holds because $\otimes_R^d F$ is a free resolution of $\otimes_R^d M$, and the equality holds because the differential on F satisfies $\partial^F \subseteq IF$. Let $\underline{e} = e_1, \dots, e_d$ be the standard basis for $F_0 = R^d$, in 3.4, and let $\underline{e}'$ be the induced basis of the free R/I-module $F_0 \otimes_R R/I$. Under the composite map, the element $\tau(\underline{e})$ maps to $\tau(\underline{e}')$, and that extends to a basis of the R/I-module $\otimes_R^d(F_0 \otimes_R R/I)$. Since $\tau(\underline{e})$ maps to $\tau(\underline{m})$ in $\otimes_R^d M$, the map in (2) splits and gives a decomposition

$$\otimes_R^d M \cong R/I \bigoplus W\,.$$

It remains to verify that W is torsion-free; given the decomposition above, the other parts of (3) are a consequence of this fact.

For $\mathfrak{p} \in \operatorname{Spec} R$ with $(\underline{r}) \not\subseteq \mathfrak{p}$, the $R_\mathfrak{p}$-module $M_\mathfrak{p}$, and hence also $W_\mathfrak{p}$, is free.

Assume $(\underline{r}) \subseteq \mathfrak{p}$. The Koszul complex on $\underline{r}$, viewed as elements in $R_\mathfrak{p}$, is a minimal resolution of $(R/I)_\mathfrak{p}$, and so it is a direct summand of $(\otimes_R^n F)_\mathfrak{p}$, the minimal free resolution of $(\otimes_R^n M)_\mathfrak{p}$. The ranks of the free modules in the top degree, d, of these complexes coincide (and equal 1), hence $\operatorname{pd}_{R_\mathfrak{p}} W_\mathfrak{p} \leq d-1$ and

$$\operatorname{depth}_{R_\mathfrak{p}} W_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} - \operatorname{pd}_{R_\mathfrak{p}} W_\mathfrak{p} \geq 1$$

Thus W is torsion-free, as claimed. □

Local rings. Next we focus on local rings, where the preceding results can be strengthened to some extent.

Lemma 3.7. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m})$, and let $m_1, \dots, m_d \in M$. If the images of $\{m_1, \dots, m_d\}$ in $M/\mathfrak{m}M$ are linearly independent, then $\tau(\underline{m})$ is not in $\mathfrak{m}(\otimes_R^d M)$.*

Proof. Let m_i' be the image of m_i in the k-vector space $M/\mathfrak{m}M$. Since $\{m_1, \dots, m_d\}$ is linearly independent, $\tau(\underline{m}') \neq 0$. Hence $\tau(\underline{m}) \notin \mathfrak{m}(\otimes_R^d M)$. □

Given an R-module M, we write $I(M)$ for the ideal (r_{ij}) defined by the entries in a matrix in some minimal presentation

$$R^\mu \xrightarrow{[r_{ij}]} R^\nu \longrightarrow M \longrightarrow 0 \quad \text{where } \nu = \nu_R(M).$$

This ideal is independent of the presentation. Moreover, it is not hard to check that $I(M)$ contains a non-zerodivisor if and only if over $\mathrm{Q}(R)$, the total quotient ring of R, the module $\mathrm{Q}(R) \otimes_R M$ can be generated by fewer than $\nu_R(M)$ elements. Recall that M is said to *have rank* r if $\mathrm{Q}(R) \otimes_R M$ is free over $\mathrm{Q}(R)$ of rank r; see [6, Proposition 1.4.3] for different characterizations of this property.

Theorem 3.8. *Let R be a local ring and M a nonzero finitely generated R-module satisfying one of the following conditions:*

(1) *$I(M)$ contains a non-zerodivisor; in this case, set $d = \nu_R(M)$; or*
(2) *M has rank; in this case, set $d = \operatorname{rank}_R(M) + 1$.*

If M is not free, then for each nonzero finitely generated R-module N one has

$$\top_R((\otimes^n_R M) \otimes_R N) \neq 0 \quad \textit{for each } n \geq d\,.$$

Proof. It suffices to prove the statement for $n = d$, since

$$(\otimes^n_R M) \otimes_R N \cong (\otimes^d_R M) \otimes_R ((\otimes^{n-d}_R M) \otimes_R N)\,,$$

and $N \neq 0$ implies $(\otimes^i_R M) \otimes_R N \neq 0$ for each $i \geq 0$, by Nakayama's lemma.

We claim that there exists a syzygy relation (3.2.1) with $\underline{m}$ a minimal generating set for M, $(\underline{r}) \subseteq \mathfrak{m}$, and some r_i a non-zerodivisor.

Indeed, when (1) holds the hypothesis on $I(M)$ implies that there is a syzygy relation as claimed. When (2) holds, $\nu_R(M) \geq d$ since M is not free. Choose elements $m_1, \dots, m_d$ that form part of a minimal generating set for M and such that $m_1, \dots, m_{d-1}$ form a basis for $\mathrm{Q}(R) \otimes_R M$ over $\mathrm{Q}(R)$. Then there is a syzygy relation as in (3.2.1) in which r_d is a non-zerodivisor.

Now, for any such syzygy relation the element $\tau(\underline{m})$ in $\otimes^d_R M$ is annihilated by $(\underline{r})$, by Theorem 3.6, and is not in $\mathfrak{m}(\otimes^d_R M)$, by Lemma 3.7. It follows that, for each n in $N \setminus \mathfrak{m}N$, the element $\tau(\underline{m}) \otimes n$ in $(\otimes^d_R M) \otimes_R N$ is nonzero and is annihilated by $(\underline{r})$, and hence is in the torsion submodule. □

Corollary 3.9. *Let R be a Noetherian domain and M a finitely generated R-module. Then the R-module $\otimes^n_R M$ has torsion for any integer n satisfying*

$$n \geq \min\left\{ \nu_{R_\mathfrak{p}}(L) \;\middle|\; \begin{array}{l} \mathfrak{p} \in \operatorname{Spec} R \textit{ and } L \textit{ is a non-free direct} \\ \textit{summand of the } R_\mathfrak{p}\textit{-module } M_\mathfrak{p} \end{array} \right\}.$$

Proof. Fix a prime $\mathfrak{p}$ where the minimum is achieved, and let L be a non-free direct summand of $M_\mathfrak{p}$ that requires at most n generators. Taking $N = R_\mathfrak{p}$ in Theorem 3.8, we see that $\otimes^n_{R_\mathfrak{p}} L$ has torsion. Since this module is a direct summand of $\otimes^n_{R_\mathfrak{p}} M_\mathfrak{p}$, the latter module must have torsion, and so must $\otimes^n_R M$, by Lemma 2.11. □

One cannot always expect torsion in tensor powers of non-free modules:

Example 3.10. Let $R = k[x,y]/(xy)$, where k is a field. The torsion-free R-module $M := R/(x)$ is not free; however $\otimes^n_R M$ is isomorphic to $R/(x)$ for every $n \geq 1$, and hence is torsion-free.

The preceding results bring to the fore the following:

Question 3.11. Let R be a local domain. Is there an integer b, depending only on R, such that $\otimes^n_R M$ has torsion for every finitely generated non-free R-module M and every integer $n \geq b$?

The condition that R be a domain is to avoid the situation of Example 3.10. When R is regular, one can take $b = \dim R$, by results of Auslander [2, Theorem 3.2] and Lichtenbaum [28, Corollary 3]. See Corollary 5.9 for a generalization.

When R is Cohen-Macaulay, such an integer b must satisfy $b \geq \dim R$. Indeed, with $\underline{x} := x_1, \dots, x_d$ a maximal regular sequence in R, the $(d-1)^{st}$ syzygy module of $R/(\underline{x})$, say M, has the following properties:

- M has rank $d-1$ but is minimally generated by d elements, so is not free.
- $\otimes_R^{d-1} M$ is torsion-free.

For the second item, see Theorem 3.6. This example also proves that the bound given in Theorem 3.8 is optimal.

Over any one-dimensional Cohen-Macaulay domain R with a non-free canonical module, Huneke and Wiegand [21, Proposition 4.7] constructed a non-free finitely generated R-module M such that $M \otimes_R M$ is torsion-free. Therefore, even if Question 3.11 has an affirmative answer, $1 + \dim(R)$ is, in general, too small.

We end this section with an easy observation characterizing those local rings for which the tensor product of torsion-free modules is *always* torsion-free.

Proposition 3.12. *Let $(R, \mathfrak{m})$ be a local ring. Then the following are equivalent:*

(1) *$M \otimes_R N$ is torsion-free whenever M, N are finitely generated torsion-free R-modules.*
(2) *$M \otimes_R M$ is torsion-free for every finitely generated torsion-free R-module M.*
(3) *$\mathfrak{m} \otimes_R \mathfrak{m}$ is torsion-free.*
(4) *Either* $\operatorname{depth}(R) = 0$ *or R is a discrete valuation domain.*

Proof. The implications (1) $\implies$ (2) $\implies$ (3) are clear. For (3) $\implies$ (4), assume R has positive depth. Then $\mathfrak{m}$ contains a non-zerodivisor and hence has rank one. If, now, $\mathfrak{m} \otimes_R \mathfrak{m}$ is torsion-free, then Theorem 3.8 (2) implies that $\mathfrak{m}$ is free, and hence principal; that is to say, that R is a discrete valuation domain. Now we show that (4) $\implies$ (1). When $\operatorname{depth}(R) = 0$ every module is torsion-free. Otherwise R is a discrete valuation domain, and the modules M and N of (1) are in fact free. Their tensor product is then free, and hence torsion-free. □

4. Complete intersections

In this section we discuss depth properties of tensor products *vis à vis* vanishing of Tor over complete intersections. Since the terminology in the literature is not entirely standard (as one might surmise from the title of [18]), let us lay out the definitions we will use. Throughout this section R is a local ring.

Definition 4.1. We write $\operatorname{embdim} R$ for the *embedding dimension* of R, that is, the minimal number of generators of the maximal ideal of R. The *codimension* of R is the number

$$\operatorname{codim} R = \operatorname{embdim} R - \dim R\,.$$

We say that R is a *complete intersection in* a local ring $(Q, \mathfrak{n})$ if there a surjection $\pi\colon Q \twoheadrightarrow R$ with $\ker \pi$ generated by a Q-regular sequence in $\mathfrak{n}$; the length of this regular sequence is the *relative codimension of R in Q*. A *hypersurface in Q* is a complete intersection of relative codimension one in Q. By a *complete intersection* (respectively, a *hypersurface*) we mean a local ring $(R, \mathfrak{m})$ whose $\mathfrak{m}$-adic completion $\widehat{R}$ is a complete intersection (respectively, a hypersurface) in a regular local ring.

Suppose R is a complete intersection in Q, of relative codimension c. One has

$$\dim Q - \dim R = c\,.$$

Also, there is an inequality

$$\operatorname{codim} R - \operatorname{codim} Q \le c\,,$$

with equality if and only if $\ker \pi \subseteq \mathfrak{n}^2$. In particular, if $\widehat{R}$ is a complete intersection of relative codimension c in a regular local ring $(Q, \mathfrak{n})$, say, $\widehat{R} = Q/(f_1, \dots, f_c)$ for a regular sequence $(\underline{f})$, then

$$\operatorname{codim} R = \operatorname{codim} \widehat{R} \leq c\,,$$

with equality if and only if $(\underline{f}) \subseteq \mathfrak{n}^2$.

Recall that a regular local ring $(Q, \mathfrak{n}, k)$ is said to be *unramified* if either Q contains a field, or else $Q \supset \mathbb{Z}$, $\operatorname{char} k = p$, and $p \notin \mathfrak{n}^2$. The regular local ring $V[x]/(x^2 - p)$, where V is the ring of p-adic integers, is *ramified*.

4.2. Ramified regular local rings. Assume $(Q, \mathfrak{n}, k)$ is a d-dimensional complete regular local ring. If Q is ramified, with $\operatorname{char} k = p$, then, for some complete unramified discrete valuation ring (V, pV), one has $Q \cong T/(p - f)$, where $T = V[[x_1, \dots, x_d]]$ and f is contained in the square of the maximal ideal of T; see for example [7, Chaper IX, §3]. Hence every complete regular local ring is a hypersurface in an unramified one. Therefore, when R is a complete intersection, $\widehat{R}$ can be realized as a complete intersection in an unramified regular local ring Q in such a way that the relative codimension c of $\widehat{R}$ in Q satisfies:

$$\operatorname{codim} R \leq c \leq \operatorname{codim} R + 1\,.$$

We will use, often implicitly, the fact [2, Lemma 3.4] that every localization, at a prime ideal, of an unramified regular local ring is again unramified.

We need a few preparatory results before getting to the main theorem of this section, namely Theorem 4.7. The first is a special case of a theorem proved by D. Jorgensen [25, Theorem 2.1]:

Remark 4.3. Let R be a complete intersection, and let M and N be finitely generated R-modules with $\operatorname{Tor}_i^R(M, N) = 0$ for $i \gg 0$. If M is maximal Cohen-Macaulay, then M and N are Tor-independent.

Lemma 4.4. *Let R be a complete intersection, and let M and N be finitely generated R-modules. If $\operatorname{Tor}_i^R(M, N)$ is torsion for all $i \gg 0$, then $\operatorname{Tor}_i^R(M, N)$ is torsion for all $i \geq 1$.*

Proof. It suffices to prove $\operatorname{Tor}_i^R(M, N)_\mathfrak{p} = 0$ for $i \geq 1$ and for each minimal prime $\mathfrak{p}$ of R, i.e., $\operatorname{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$ for $i \geq 1$. For $i \gg 0$, we have $\operatorname{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$. If $M_\mathfrak{p} = 0$, we're done. Otherwise, $M_\mathfrak{p}$ is MCM since $\dim R_\mathfrak{p} = 0$, and Remark 4.3 gives the desired vanishing. □

In our applications, we will encounter the same hypotheses enough times to warrant a piece of notation.

Notation 4.5. Let c be a positive integer. We will say that a pair M, N of modules over a ring R satisfies (SP_c) provided the following conditions hold:

(1) M and N satisfy (S_{c-1}).
(2) $M \otimes_R N$ satisfies (S_c).
(3) $\operatorname{Tor}_i^R(M, N)$ has finite length for all $i \gg 0$.

The (SP_c) condition imposes restrictions on the support of the Tors:

Lemma 4.6. *Let* $(R, \mathfrak{m})$ *be a complete intersection, and let* M *and* N *be finitely generated* R*-modules. Assume that* M *satisfies* (S_{c-1}) *for some integer* $c \geq 1$ *and that* $\operatorname{Tor}_i^R(M, N)$ *has finite length for all* $i \gg 0$*. Let* $\mathfrak{p}$ *be a non-maximal prime ideal with* $\operatorname{height} \mathfrak{p} \leq c-1$*. Then* $\operatorname{Tor}_i^R(M, N)_{\mathfrak{p}} = 0$ *for all* $i \geq 1$*.*

Proof. The Serre condition implies that $M_{\mathfrak{p}}$ is either 0 or MCM. Also, we have $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for $i \gg 0$ because $\mathfrak{p} \neq \mathfrak{m}$. Therefore $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for all $i \geq 1$ by Remark 4.3 . □

The next theorem generalizes a result due to Celikbas [9, 3.16]; we emphasize that the ambient regular local ring in Theorem 4.7 is allowed to be ramified.

Theorem 4.7. *Let* R *be a complete intersection with* $\dim R \geq \operatorname{codim} R$*, and let* M *and* N *be finitely generated* R*-modules. Assume the pair* M, N *satisfies* (SP_c) *for some* $c \geq \operatorname{codim} R$*. If* $c = 1$*, assume further that* M *or* N *is torsion-free. If* $\operatorname{Tor}_1^R(M, N) = 0$*, then* $\operatorname{Tor}_i^R(M, N) = 0$ *for all* $i \geq 1$*.*

Proof. Without loss of generality, one may assume $c = \operatorname{codim} R$. When $c = 0$, the desired result is the rigidity theorem of Auslander [2] and Lichtenbaum [28], so in the remainder of the proof we assume that $c \geq 1$. When $c = 1$, assume N is the module that is torsion-free.

We define a sequence of modules $M_0, M_1, \ldots, M_{c-1}$ by setting $M_0 = M$ and then M_n to be the pushforward of M_{n-1}, for $n = 1, \ldots, c-1$. These pushforwards exist because M_0 satisfies (S_{c-1}) by hypothesis, and so each M_{n-1} satisfies (S_{c-n}) by Proposition 2.8. For the desired result, it suffices to prove $\operatorname{Tor}_i^R(M_{c-1}, N) = 0$ for each $i \geq c$. We will, in fact, prove this for all $i \geq 1$. To this end, we claim that, for $n = 0, \ldots, c-1$, the following hold:

(1) M_n satisfies (S_{c-1-n});
(2) $M_n \otimes_R N$ satisfies (S_{c-n});
(3) $\operatorname{Tor}_i^R(M_n, N)$ has finite length for all $i \gg 0$;
(4) $\operatorname{Tor}_i^R(M_n, N) = 0$ for $i = 1, \ldots, n+1$.

For $n = 0$, the first three are from the (SP_c) conditions and (4) holds by hypothesis. Assume they hold for some integer n with $0 \leq n \leq c-2$; we then verify that they hold also for $n+1$. As noted before, (1) holds by Proposition 2.8. Consider the pushforward

$$0 \longrightarrow M_n \longrightarrow F \longrightarrow M_{n+1} \longrightarrow 0\,;$$

here F is free. Tensoring with N yields isomorphisms and an exact sequence:

$$\operatorname{Tor}_i^R(M_{n+1}, N) \cong \operatorname{Tor}_{i-1}^R(M_n, N) \quad \text{for all } i \geq 2 \tag{4.7.1}$$

$$0 \longrightarrow \operatorname{Tor}_1^R(M_{n+1}, N) \longrightarrow M_n \otimes_R N \longrightarrow F \otimes_R N \longrightarrow M_{n+1} \otimes_R N \longrightarrow 0 \tag{4.7.2}$$

The isomorphisms (4.7.1) imply that $\operatorname{Tor}_i^R(M_{n+1}, N)$ has finite length for all $i \gg 0$, so (3) holds; also, these modules are torsion for each $i \geq 1$, by Lemma 4.4. (Note that $\dim(R) \geq \operatorname{codim}(R) = c \geq 1$, so finite-length modules are torsion.) Since $n \leq c-2$, condition (2) implies that $M_n \otimes_R N$ satisfies (S_2) and hence is torsion-free; therefore the exact sequence (4.7.2) yields $\operatorname{Tor}_1^R(M_{n+1}, N) = 0$. This, with the isomorphisms in (4.7.1), gives (4).

It remains to verify (2), that $M_{n+1} \otimes_R N$ satisfies (S_{c-n-1}).

Since N satisfies (S_{c-1}) and hence (S_{c-n-1}), so does $F \otimes_R N$. Also, (4.7.2) is actually a short exact sequence since $\operatorname{Tor}_1^R(M_{n+1}, N) = 0$. When we localize this

exact sequence at a prime $\mathfrak{p}$ with height $\mathfrak{p} \geq c-n$, we have $\operatorname{depth}_{R_\mathfrak{p}}(M_n \otimes_R N)_\mathfrak{p} \geq c-n$, by (2) of the induction hypothesis, and $\operatorname{depth}_{R_\mathfrak{p}}(F \otimes_R N)_\mathfrak{p} \geq c-n-1$. By the depth lemma [6, Proposition 1.2.9], one has $\operatorname{depth}_{R_\mathfrak{p}}(M_{n+1} \otimes_R N)_\mathfrak{p} \geq c-n-1$.

Now fix a prime $\mathfrak{p}$ in $\mathbb{X}^{c-n-1}(R)$; we want to show that $(M_{n+1} \otimes_R N)_\mathfrak{p}$ is MCM or zero. We may suppose the $R_\mathfrak{p}$-modules $N_\mathfrak{p}$ and $(M_n)_\mathfrak{p}$ are nonzero and hence MCM; from the latter we conclude that $(M_{n+1})_\mathfrak{p}$ is MCM or zero as well; see Proposition 2.8. We may assume it is MCM. The pair M_n, N satisfies (SP_{c-n}), so

$$\operatorname{Tor}_i^R(M_n, N)_\mathfrak{p} = 0 \quad \text{for } i \geq 1$$

by Lemma 4.6; note that $\mathfrak{p} \neq \mathfrak{m}$ for $\dim R \geq \operatorname{codim} R = c > c-n-1$, as $n \geq 0$. The isomorphisms in (4.7.1) and the already established equality $\operatorname{Tor}_1^R(M_{n+1}, N) = 0$ then give $\operatorname{Tor}_i^R(M_{n+1}, N)_\mathfrak{p} = 0$ for $i \geq 1$. The depth formula (1.0.1) now shows that $\operatorname{depth}_{R_\mathfrak{p}}(M_{n+1} \otimes_R N)_\mathfrak{p} = \dim R_\mathfrak{p}$. We have shown that $M_{n+1} \otimes_R N$ satisfies (S_{c-n-1}), and the proof of the claim is now complete.

At the end, $\operatorname{Tor}_i^R(M_{c-1}, N)$ has finite length for all $i \gg 0$ and is equal to 0 for $i = 1, \ldots, c$; moreover, $M_{c-1} \otimes_R N$ is torsion-free. Now N satisfies (S_{c-1}) and hence is torsion-free; recall that when $c = 1$ we have assumed N is torsion-free. Tensoring M_{c-1} with the pushforward for N, we thus get the following isomorphisms and exact sequence:

$$\operatorname{Tor}_i^R(M_{c-1}, N_1) \cong \operatorname{Tor}_{i-1}^R(M_{c-1}, N) = 0 \quad \text{for } i = 2, \ldots, c+1$$

$$0 \longrightarrow \operatorname{Tor}_1^R(M_{c-1}, N_1) \longrightarrow M_{c-1} \otimes_R N \longrightarrow M_{c-1} \otimes_R G \longrightarrow M_{c-1} \otimes_R N_1 \longrightarrow 0$$

By the same argument as before (using Lemma 4.4), $\operatorname{Tor}_1^R(M_{c-1}, N_1) = 0$ as well. We now have $\operatorname{Tor}_i^R(M_{c-1}, N_1) = 0$ for $i = 1, \ldots, c+1$. Since R is a complete intersection of codimension c, it follows from Murthy's rigidity theorem [33, 1.6] that $\operatorname{Tor}_i^R(M_{c-1}, N_1) = 0$ for all $i \geq 1$. Shifting along the pushforward for N, we see that $\operatorname{Tor}_i^R(M_{c-1}, N) = 0$ for all $i \geq 1$. □

Note that the proof of the preceding theorem is much simpler when $c = 1$. This case is all one needs to prove the following corollary, due to Huneke and Wiegand [21, Theorem 2.7]; see also [11, 22].

Corollary 4.8. *Let R be a hypersurface and M, N nonzero finitely generated R-modules. Assume $M \otimes_R N$ is reflexive and N has rank. Then:*

(1) $\operatorname{Tor}_i^R(M, N) = 0$ *for* $i \geq 1$.
(2) *M is reflexive, and N is torsion-free.*

Proof. We remark at the outset that neither M nor N can be torsion, that is $\top_R M \neq M$ and $\top_R N \neq N$; also, $\operatorname{Supp}(N) = \operatorname{Spec}(R)$. Suppose first that both M and N are torsion-free; we will prove (1) by induction on $d := \dim R$. If $d = 0$, then N is free, so all is well. Assume $d \geq 1$. When we tensor the pushforward (2.7.1)

$$0 \to M \to R^{(\nu)} \to M_1 \to 0$$

with N, we obtain an exact sequence

$$0 \to \operatorname{Tor}_1^R(M_1, N) \to M \otimes_R N \to N^{(\nu)} \to M_1 \otimes_R N \to 0\,.$$

Now $\operatorname{Tor}_1^R(M_1, N)$ is a torsion module because N has rank. Since $M \otimes_R N$ is torsion-free, $\operatorname{Tor}_1^R(M_1, N) = 0$. The inductive hypothesis implies that $\operatorname{Tor}_i^R(M, N)$ has finite length for all $i \geq 1$, and from the pushforward exact sequence we see that

$\operatorname{Tor}_i^R(M_1, N)$ has finite length for $i \geq 2$. We want to use Theorem 4.7 to conclude that $\operatorname{Tor}_i^R(M_1, N) = 0$ for all $i \geq 1$, but for that we need to know that $M_1 \otimes_R N$ is torsion-free, that is, that it satisfies (S_1). To that end, we let $\mathfrak{p}$ be a prime ideal with $\operatorname{height}(\mathfrak{p}) \geq 1$, and localize the exact sequence

$$0 \to M \otimes_R N \to N^{(\nu)} \to M_1 \otimes_R N \to 0$$

at $\mathfrak{p}$. If $\operatorname{height}(\mathfrak{p}) \geq 2$, the facts that $\operatorname{depth}(M \otimes_R N)_{\mathfrak{p}} \geq 2$ and $\operatorname{depth} N_{\mathfrak{p}} \geq 1$ imply that $\operatorname{depth}(M_1 \otimes_R N)_{\mathfrak{p}} \geq 1$.

Suppose $\operatorname{height}(\mathfrak{p}) = 1$. Arguing as in the first part of the proof of [21, Theorem 3.7], we find that $\operatorname{Tor}_i^R(M, N)_{\mathfrak{p}} = 0$ for all $i \geq 1$. Since $\operatorname{Tor}_1^R(M_1, N)_{\mathfrak{p}} = 0$, the pushforward exact sequence shows that $\operatorname{Tor}_i^R(M_1, N)_{\mathfrak{p}} = 0$ for all $i \geq 1$. Now $N_{\mathfrak{p}}$ and $(M_1)_{\mathfrak{p}}$ are MCM, the latter by Proposition 2.8(2). The depth formula (1.0.1) now implies that $(M_1 \otimes_R N)_{\mathfrak{p}}$ is MCM as well. This completes the proof that $M_1 \otimes_R N$ is torsion-free, and now Theorem 4.7 implies that $\operatorname{Tor}_i^R(M_1, N) = 0$ for all $i \geq 1$. Shifting along the pushforward gives $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. This proves (1) under the additional assumption that M and N are torsion-free.

We now relax that assumption and complete the proof using a standard argument going back to [2, Lemma 3.1] (cf. [21, Lemma 1.1]). We tensor the exact sequence (2.3.1) with N, getting an exact sequence

$$(\top_R M) \otimes_R N \xrightarrow{\alpha} M \otimes_R N \xrightarrow{\beta} (\perp_R M) \otimes_R N \to 0\,.$$

Since $(\top_R M) \otimes_R N$ is torsion and $M \otimes_R N$ is torsion-free, the map α must be zero, and hence β is an isomorphism. In particular, $(\perp_R M) \otimes_R N$ is reflexive as well. Tensoring $\perp_R M$ with the exact sequence (2.3.1) for N, we get an exact sequence

$$\operatorname{Tor}_1^R(\perp M, \perp N) \to (\perp M) \otimes_R (\top N) \to (\perp M) \otimes_R N \to (\perp M) \otimes_R (\perp N) \to 0\,,$$

and we learn as before that $(\perp_R M) \otimes_R (\perp_R N)$ is reflexive. By the first part of the proof, $\operatorname{Tor}_i^R(\perp_R M, \perp_R N) = 0$ for all $i \geq 1$. The vanishing of $\operatorname{Tor}_1^R(\perp_R M, \perp_R N)$ and the exact sequence above show that $(\perp_R M) \otimes_R (\top_R N) = 0$, whence $\top_R N = 0$, that is, $\perp_R N = N$. The same steps, with the order of M and N reversed, show that $\perp_R M = M$, and hence that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

The remaining step is to prove that M is reflexive. Since $\operatorname{Supp}(N) = \operatorname{Spec}(R)$, we have $\operatorname{depth} N_{\mathfrak{p}} \leq \operatorname{height} \mathfrak{p}$ for each prime ideal $\mathfrak{p}$. Now one easily verifies Serre's condition (S_2) on M by localizing the depth formula (1.0.1). □

Remark 4.9. If R is a hypersurface and $\operatorname{Tor}_i^R(M, N) = 0$ for $i \gg 0$, then either M or N has finite projective dimension.

This is [23, Theorem 1.9]. The proof there used the main result (Corollary 4.10 below) of [21], but C. Miller [30, Theorem 1.1] gave a more direct argument.

The next corollary, due to Huneke and Wiegand, is [21, Theorem 3.1], the main theorem of [21]. We will deduce it from Corollary 4.8 and Remark 4.9; see Remarks 4.11 for further comments.

Corollary 4.10. *Let R be a hypersurface and M, N nonzero finitely generated R-modules. Suppose $M \otimes_R N$ is MCM and either M or N has rank. Then both M and N are MCM, and one of them is free.*

Proof. As $M \otimes_R N$ satisfies (S_2) Corollary 4.8 implies $\operatorname{Tor}_i^R(M, N) = 0$ for $i \geq 1$. The depth formula (1.0.1) then shows that both M and N must be MCM. Moreover,

one of the modules has finite projective dimension by Remark 4.9, and hence is free by the Auslander-Buchsbaum formula [3, Theorem 3.7]. □

Remarks 4.11. The argument that proves Corollary 4.10 using Corollary 4.8 is exactly the one given by C. Miller [30, Theorem 1.4], who however omits the hypothesis that M or N has rank; the result is false without that hypothesis, as Example 3.10 shows.

With this correction, one can deduce, as is done in [30, Theorem 3.1], the following result from Corollary 4.10: *Let R be a hypersurface and M a finitely generated R-module with rank. If $\otimes_R^n M$ is reflexive for some integer $n \geq \dim R - 1$, then M is free*; see Corollary 5.9 for a related statement.

5. The η-pairing

The main tool in this section is the η-pairing, introduced by Dao [13]. We start with the definition and basic properties of the η-pairing. We focus on complete intersections, as all of our applications are in that context.

5.1. The η-pairing. Let R be a complete intersection ring and let M and N be finitely generated R-modules. Assume there exists an integer l such that $\operatorname{Tor}_i^R(M,N)$ has finite length for all $i \geq l$. For each positive integer e, the pairing $\eta_e^R(M,N)$ is defined as follows:

$$\eta_e^R(M,N) = \lim_{n\to\infty} \frac{1}{n^e} \sum_{i=l}^{n} (-1)^i \operatorname{length}_R \operatorname{Tor}_i^R(M,N)\,.$$

Clearly $\eta_e^R(M,N) = \eta_e^R(N,M)$. The function $\eta_e^R(M,-)$ is additive on short exact sequences, provided η_e^R is defined on each term. The following statements hold:

(1) $\eta_e^R(M,N) = 0$ if $e > \operatorname{codim} R$.
(2) $\eta_e^R(M,N) < \infty$ if $e = \operatorname{codim} R$.
(3) $\eta_e^R(M,N) = 0$ if either M or N has finite projective dimension.
(4) $\eta_e^R(M,N) = 0$ if either M or N has finite length and $e \geq \operatorname{codim} R$.

These results follow from [13, 4.3], and will be used without further ado.

The η-pairing is a natural extension to complete intersections of the θ-pairing over hypersurfaces, introduced by Hochster [19] in 1980. For any hypersurface R and finitely generated R-modules M and N there is an equality:

$$\eta_1^R(M,N) = \frac{1}{2}\,\theta^R(M,N)\,.$$

In particular, $\eta_1^R(M,N) = 0$ if and only if $\theta^R(M,N) = 0$.

For the present purposes, the utility of the η-pairing stems from the following result of Dao [13, Theorem 6.3] that its vanishing implies a version of the rigidity of Tor: If c consecutive Tors vanish, then so do all subsequent ones.

Proposition 5.2. *Let R be a local ring whose completion is a complete intersection, of relative codimension $c \geq 1$, in an unramified regular local ring. Let M, N be finitely generated R-modules. Assume that $\operatorname{Tor}_i^R(M,N)$ has finite length for all $i \gg 0$ and that $\eta_c^R(M,N) = 0$. If $\operatorname{Tor}_i^R(M,N) = 0$ for $i = s, \dots, s+c-1$, where $s \geq 0$, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq s$.*

Proof. By hypothesis, $\widehat{R} \cong Q/I$, where $(Q, \mathfrak{n})$ is an unramified regular local ring and I is generated by a Q-regular sequence of length c; in particular, R is a complete intersection. When $I \subseteq \mathfrak{n}^2$ (the main case of interest), the desired statement is [13, Theorem 6.3]. When $I \not\subseteq \mathfrak{n}^2$ the codimension of R is at most $c-1$ and Murthy's theorem [33, Theorem 1.6] gives the stated vanishing. In this case the condition $\eta_c^R(M,N) = 0$ is automatic. □

Theorem 5.3. *Let R be a local ring whose completion is a hypersurface in an unramified regular local ring, with $\dim R \geq 1$. Let M, N be a pair of finitely generated R-modules such that $M \otimes_R N$ is torsion-free.*

(1) *$\eta_1^R(M, \perp_R N)$ is defined and equal to 0, and $M \neq 0$; or*
(2) *$\eta_1^R(M, N)$ is defined and equal to 0, and $\operatorname{Supp}_R \top_R N \subseteq \operatorname{Supp}_R M$.*

Then M and N are Tor-independent and N is torsion-free.

Remarks 5.4. Parts (1) and (2) are the same statement, when N is torsion-free. The condition on supports in (2) holds when, for example, the support of N is contained in that of M. Moreover, if R is a domain and M and N are nonzero, it is a consequence of the torsion-freeness of $M \otimes_R N$, for then the support of $M \otimes_R N$, and hence also of M, is all of $\operatorname{Spec} R$.

Proof of Theorem 5.3. Suppose (1) holds, and apply $M \otimes_R -$ to the exact sequence

$$0 \to \top_R N \to N \to \perp_R N \to 0\,. \tag{5.4.1}$$

Since $M \otimes_R \top_R N$ is torsion and $M \otimes_R N$ is torsion-free, the map between them is zero. Thus $M \otimes_R N$ is isomorphic to $M \otimes_R \perp_R N$ and hence the latter module is torsion-free too.

Next, we build the pushforward

$$0 \to \perp_R N \to F \to Z \to 0\,. \tag{5.4.2}$$

By hypothesis, $\operatorname{Tor}_i^R(M, \perp_R N)$ has finite length for $i \gg 0$, and we see from (5.4.2) that $\operatorname{Tor}_i^R(M, Z)$ has finite length for $i \gg 0$. Since $\dim R \geq 1$ it follows that $\operatorname{Tor}_i^R(M, Z)$ is torsion for all $i \gg 0$ and hence for all $i \geq 1$, by Lemma 4.4. Applying $M \otimes_R -$ to (5.4.2), we get an injection from the torsion module $\operatorname{Tor}_1^R(M, Z)$ to the torsion-free module $M \otimes_R \perp_R N$ and conclude that $\operatorname{Tor}_1^R(M, Z) = 0$. Additivity of η_1^R along the exact sequence (5.4.2) yields $\eta_1^R(M, Z) = 0$, and Proposition 5.2 implies that $\operatorname{Tor}_i^R(M, Z) = 0$ for all $i \geq 1$.

Shifting along (5.4.2), we have

$$\operatorname{Tor}_i^R(M, \perp_R N) = 0 \text{ for all } i \geq 1\,. \tag{5.4.3}$$

Once again applying $M \otimes_R (-)$ to (5.4.1) gives an injection $M \otimes_R \top_R N \hookrightarrow M \otimes_R N$, so $M \otimes_R \top_R N = 0$. Since $M \neq 0$, this implies that $\top_R N = 0$, and then (5.4.3) tells us that $\operatorname{Tor}_i^R(M, N) = 0$ for each $i \geq 1$.

Suppose now that (2) holds. We may assume that $M \neq 0$. Showing $\top_R N = 0$ will return us to case (1). Supposing that $\top_R N \neq 0$, let $\mathfrak{p}$ be a prime minimal in $\operatorname{Supp}_R(\top_R N)$. Then $(\top_R N)_{\mathfrak{p}}$ is a nonzero $R_{\mathfrak{p}}$-module of finite length, and, moreover, $\dim R_{\mathfrak{p}} \geq 1$ since $(\top_R N)_{\mathfrak{q}} = 0$ if $\mathfrak{q} \in \operatorname{Ass}(R)$.

Note that $(\top_R N)_{\mathfrak{p}}$ is a torsion $R_{\mathfrak{p}}$-module and $(\perp_R N)_{\mathfrak{p}}$ is torsionless, hence torsion-free, as an $R_{\mathfrak{p}}$-module. It follows that $(\top_R N)_{\mathfrak{p}}$ is exactly the torsion submodule of $N_{\mathfrak{p}}$. Now $\eta_1^{R_{\mathfrak{p}}}$ vanishes on modules of finite length (item (4) of 5.1). By additivity of $\eta_1^{R_{\mathfrak{p}}}$ along the exact sequence $(5.4.1)_{\mathfrak{p}}$, we have $\eta_1^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (\perp_R N)_{\mathfrak{p}}) = 0$.

Moreover, $M_\mathfrak{p} \neq 0$, by the assumption on supports. We are now in case (1), and we conclude that $(\top_R N)_\mathfrak{p} = 0$, contradicting the choice of $\mathfrak{p}$. $\square$

Example 5.5. Most of the hypotheses in Theorem 5.3 are essential; see the discussion after [23, Remark 1.5]. To begin with, without the assumption that $\dim R \geq 1$, the theorem would fail. Take, for example, $R = k[x]/(x^2)$ and $M = R/(x) = N$.

The vanishing of η is also essential: Let $R = k[[x, y]]/(xy)$, where k is any field, and put $M = R/(x)$ and $N = R/(x^2)$. Then $R/(x)$ is free at the minimal primes—but not of constant rank—and $M \otimes_R N$ is just M, which is torsion-free. Thus the pair M, N satisfies (SP_1). On the other hand, x is a nonzero torsion element of N, since it is killed by the non-zerodivisor $x + y$. Therefore the depth formula (1.0.1) fails, and hence M and N cannot be Tor-independent. In fact, one can check that $\operatorname{Tor}^R_{2i+1}(M, N) \cong k$ for $i \geq 0$ and that the even ones vanish, so $\eta^R_1(M, N) = -\frac{1}{2}$.

We do not know if in Theorem 5.3 the assumption on supports is superfluous. In this context, we remind the reader of the following open question, implicit in [21]:

Question 5.6. Let R be a complete intersection and M, N finitely generated R-modules. If $M \otimes_R N$ is torsion-free, then must one of M, N be torsion-free?

For Gorenstein domains, the answer to Question 5.6 is "no". Indeed, for the ring $R = k[\![t^8, \ldots, t^{14}]\!]$, P. Constapel [12] built modules M and N, neither torsion-free, such that $M \otimes_R N$ is torsion-free.

Next we present a variant of Theorem 5.3.

Remark 5.7. Suppose that the finitely generated module M has finite projective dimension on the punctured spectrum of R, meaning that, for $d = \dim R$, one has

$$\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty \quad \text{for all } \mathfrak{p} \in \mathbb{X}^{d-1}(R).$$

Then, for any finitely generated R-module N, $\operatorname{Tor}^R_i(M, N)$ has finite length for $i \gg 0$, so the function $\eta^R_c(M, -)$ is defined; moreover, it is finite for $c \geq \operatorname{codim} R$.

The hypotheses of the next theorem imply that the pair M, N satisfies (SP_1) and that $\eta^R_1(M, \perp_R N) = 0$. The theorem is thus an immediate consequence of part (1) of Theorem 5.3. The hypothesis of finite projective dimension on the punctured spectrum is included for clarity; it is a consequence of the assumption that $\eta_1(M, N)$ is defined for every N.

Theorem 5.8. *Let R be a local ring whose completion is a hypersurface in an unramified regular local ring, and assume $\dim R \geq 1$. Let M, N be nonzero R-modules such that $M \otimes_R N$ is torsion-free. Assume that the projective dimension of M is finite on the punctured spectrum of R and that $\eta^R_1(M, -) = 0$. Then M and N are Tor-independent and N is torsion-free.* $\square$

Since the completion of any regular ring is a hypersurface in an unramified regular local ring (see 4.2), the following corollary extends [28, Corollary 3], which in turn builds on [2, Theorem 3.2]:

Corollary 5.9. *Let R be a local ring whose completion is a hypersurface in an unramified regular local ring. Assume $\dim R \geq 1$, and let M be an R-module with finite projective dimension on the punctured spectrum of R and satisfying $\eta^R_1(M, -) = 0$.*

If $\otimes^n_R M$ is torsion-free for some integer $n \geq 2$, then

$$\operatorname{pd}_R M \leq (\dim R - 1)/n\,.$$

Consequently, if M is not free, then $\otimes^n_R M$ has torsion for each $n \geq \max\{2, \dim R\}$.

Proof. We may assume $M \neq 0$. For $p = 1, \ldots, n$, repeated application of Theorem 5.8 shows that $\otimes_R^p M$ is torsion-free, and also that

$$\operatorname{Tor}_i^R(M, \otimes_R^{p-1} M) = 0 \quad \text{for } i \geq 1\,.$$

Taking $p = 2$, we see from Remark 4.9 that M has finite projective dimension. Then from [2, Corollary 1.3] one obtains the first equality below:

$$n \operatorname{pd}_R M = \operatorname{pd}_R(\otimes_R^n M) = \dim R - \operatorname{depth}_R(\otimes_R^n M) \leq \dim R - 1\,.$$

The second equality is by the Auslander-Buchsbaum formula [3, Theorem 3.7], while the inequality holds because $\otimes_R^n M$ is torsion-free. □

Next we extend the preceding results to rings of codimension ≥ 2; see Theorem 5.11. The result below is the inductive step in its proof.

Proposition 5.10. *Let $(S, \mathfrak{n})$ be a complete intersection and R a hypersurface in S. Let M and N be finitely generated torsion-free R-modules such that $\operatorname{Tor}_i^R(M,N)$ has finite length for $i \gg 0$. Let E and F be the quasi-liftings of M and N, respectively, to S. Then $\operatorname{Tor}_i^S(E,F)$ has finite length for $i \gg 0$, and*

$$\eta_{e-1}^S(E,F) = 2e\eta_e^R(M,N) \quad \textit{when } e \geq \max\{2, \operatorname{codim} S + 1\}.$$

In particular, $\eta_{e-1}^S(E,F) = 0 \iff \eta_e^R(M,N) = 0$.

Proof. By hypothesis, $R \cong S/(f)$ where f is a non-zerodivisor in S. The spectral sequence associated to the change of rings $S \to R$ yields an exact sequence

$$\cdots \to \operatorname{Tor}_{n-1}^R(M,N) \to \operatorname{Tor}_n^S(M,N) \to \operatorname{Tor}_n^R(M,N) \to \cdots \qquad (n \geq 1)\,.$$

See [28, pp. 223–224], or [33, p. 561]. This implies that $\operatorname{Tor}_i^S(M,N)$ has finite length for $i \gg 0$. Let M_1 and N_1 be the pushforwards of M and N, respectively. Since $\operatorname{Tor}_i^S(R,-) = 0$ for $i \geq 2$, there are isomorphisms

$$\operatorname{Tor}_i^S(E,N) \cong \operatorname{Tor}_{i+1}^S(M_1,N) \cong \operatorname{Tor}_i^S(M,N) \quad \text{for } i \geq 2,$$

where the one on the left comes from (2.7.2) and that on the right is from (2.7.1). Arguing in the same vein, one gets isomorphisms

$$\operatorname{Tor}_i^S(E,F) \cong \operatorname{Tor}_i^S(E,N) \quad \text{for } i \geq 2.$$

It follows that the length of $\operatorname{Tor}_i^S(E,F)$ is finite for $i \gg 0$, as desired.

At this point we know that the η-pairing is defined on (M,N), over R and also over S. Similar arguments show that the η-pairing is defined also for all pairs (X,Y) with $X \in \{M, M_1, E\}$ and $Y \in \{N, N_1, F\}$.

Now $\operatorname{codim} S \leq e-1$ by hypothesis, and hence $\operatorname{codim} R \leq e$. Therefore $\eta_e^R(-,-)$ and $\eta_{e-1}^S(-,-)$ are finite whenever they are defined. The additivity of η along the exact sequences (2.7.1) and (2.7.2) thus gives equalities

$$\eta_e^R(M,N) = -\eta_e^R(M_1,N) = \eta_e^R(M_1,N_1)\,,$$

and also the following ones:

$$\eta_{e-1}^S(E,F) = -\eta_{e-1}^S(M_1,F) = \eta_{e-1}^S(M_1,N_1)\,.$$

Our assumption that $e \geq \max\{2, \operatorname{codim} S + 1\}$, together with [13, Theorem 4.1 (3)], allow us to invoke [13, Theorem 4.3 (3)], which says that

$$\eta_e^R(M_1,N_1) = \frac{1}{2e}\eta_{e-1}^S(M_1,N_1)\,.$$

□

The next result uses the (SP_c) condition introduced in Notation 4.5. It extends of Theorem 5.3(2) to higher codimensions. Note that when $c \geq 2$, the (SP_c) condition ensures that N is torsion-free, so the condition on supports is automatic.

Theorem 5.11. *Let R be a local ring whose completion is a complete intersection in an unramified regular local ring, of relative codimension $c \geq 1$. Assume $\dim R \geq c$, and let M and N be finitely generated R-modules satisfying (SP_c) and with $\operatorname{Supp}_R \top_R N \subseteq \operatorname{Supp}_R M$.*

If $\eta_c^R(M,N) = 0$, then $\operatorname{Tor}_i^R(M,N) = 0$ for $i \geq 1$.

Proof. The case $c = 1$ is Theorem 5.3. Assume now that $c \geq 2$, and proceed by induction on c. Since $\dim R \geq c$, Lemma 4.6 yields

$$\operatorname{Tor}_i^R(M,N)_{\mathfrak{p}} = 0 \quad \text{for } i \geq 1 \text{ and each } \mathfrak{p} \in \mathbb{X}^{c-1}(R). \tag{5.11.1}$$

We can assume that R is complete. Then $R = Q/(\underline{f})$, where Q is an unramified regular local ring and $\underline{f} = f_1, \ldots, f_c$ is a Q-regular sequence. Set $(S, \mathfrak{n}) = Q/(f_1, \ldots, f_{c-1})$ and $f = f_c$.

Note that M and N are torsion-free, since they satisfy (SP_c); therefore we can construct their quasi-liftings, E and F, to S. Using the vanishing of Tors in (5.11.1) and [20, Theorem 1.8], we see that

$$E \otimes_S F \text{ satisfies } (S_{c-1})\,. \tag{5.11.2}$$

By [20, Propositions 1.6, 1.7], the assumptions in (1) of (SP_c) pass to E and F as

$$E \text{ and } F \text{ satisfy } (S_{c-1})\,. \tag{5.11.3}$$

Proposition 5.10 guarantees that $\operatorname{Tor}_i^S(E,F)$ has finite length for $i \gg 0$ (so the pair E, F satisfies (SP_{c-1})) and also that $\eta_{c-1}(E,F) = 0$. Moreover, E and F, being syzygies, are torsion-free, so we indeed have $\operatorname{Supp}_S(\top_S F) \subseteq \operatorname{Supp}_S E$. The inductive hypothesis now implies that

$$\operatorname{Tor}_i^S(E,F) = 0\,. \tag{5.11.4}$$

Given (5.11.1) and that by Notation 4.5(2) the R-module $M \otimes_R N$ is reflexive, we see from (5.11.4) and [9, Proposition 3.2 (3)] that $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. $\square$

The following result was proved in [13, 7.6] under the additional hypothesis that the completion of R is a complete intersection, of relative codimension e, in an unramified regular local ring.

Corollary 5.12. *Let R be a complete intersection and M, N finitely generated R-modules. Suppose there exists an integer $e \geq \operatorname{codim} R$ such that*

(1) *M is free on $\mathbb{X}^e(R)$,*
(2) *M and N satisfy (S_e), and*
(3) *$M \otimes_R N$ satisfies (S_{e+1}).*

Then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$.

Proof. If $e = 0$ this is the theorem of Auslander [2] and Lichtenbaum [28, Corollary 2]; see page 1 here. Assume now that $e \geq 1$. We use induction on $\dim R$. If $\dim R \leq e$, condition (1) implies that M is free, and there is nothing to prove. Assuming $\dim R \geq e+1$, we note that the hypotheses localize, so $\operatorname{Tor}_i^R(M,N)_{\mathfrak{p}} = 0$ for each $i \geq 1$ and each prime ideal $\mathfrak{p}$ in the punctured spectrum of R; that is to say, $\operatorname{Tor}_i^R(M,N)$ has finite length for all $i \geq 1$. Thus the pair M, N satisfies (SP_{e+1}). Moreover, since $\operatorname{codim} R < e+1$, the completion of R can be realized as a complete

intersection, of relative codimension $e+1$, in an unramified regular local ring (see 4.2). Hence the desired result follows from Theorem 5.11. $\square$

5.13. Isolated singularities. A local ring R is an *isolated singularity* if for each non-maximal prime ideal $\mathfrak{p}$ the ring $R_{\mathfrak{p}}$ is regular.

Over such a ring, each module is of finite projective dimension on the punctured spectrum, so $\operatorname{Tor}_i^R(M,N)$ has finite length for all $i \gg 0$ and for all finitely generated R-modules M and N; hence the pairing $\eta_c^R(M,N)$ is defined.

Corollary 5.14. *Let R be a local ring whose completion is a complete intersection in an unramified regular local ring, of relative codimension c. Assume $\dim R \geq c$ and that R is an isolated singularity. Let M and N be MCM R-modules.*

When $\eta_c^R(M,N)=0$ the following conditions are equivalent:

(1) *$M \otimes_R N$ is MCM.*
(2) *$M \otimes_R N$ satisfies (S_c).*
(3) *M and N are Tor-independent.*

Proof. Evidently (1) $\Longrightarrow$ (2), while (3) $\Longrightarrow$ (1) by the depth formula (1.0.1). Finally, (2) $\Longrightarrow$ (3) by Theorem 5.11. $\square$

6. Vanishing of the η-pairing

The critical hypothesis in the results in Section 5 is that $\eta_c^R(M,N)=0$ for a pair of modules M,N. It is not easy to verify this condition, unless the projective dimension of M or N is finite, when the pairing vanishes for trivial reasons. Next we describe classes of rings for which it is known or conjectured that $\eta^R(-,-)=0$. These rings are all isolated singularities, as defined in 5.13.

6.1. Torsion in the reduced Grothendieck group. The *Grothendieck group* $G(R)$ of a ring R is the abelian group F/S, where F is the free abelian group generated by isomorphism classes $[X]$ of finitely generated R-modules, and S is the subgroup generated by all elements of the form $[X']+[X'']-[X]$ for each exact sequence $0 \to X' \to X \to X'' \to 0$ of R-modules. The *reduced* Grothendieck group $\underline{G}(R)$ is $G(R)/L$, where L is the subgroup generated by classes of modules of finite projective dimension.

Assume R is a complete intersection, and that M has finite projective dimension on the punctured spectrum. Then, for each integer $c \geq \operatorname{codim} R$, the pairing $\eta_c^R(M,N)$ is defined and is finite; see Remark 5.7. Since the η-pairing is additive on exact sequences and vanishes when N has finite projective dimension, it induces a homomorphism $\eta_c^R(M,-)\colon \underline{G}(R) \to \mathbb{R}$ of abelian groups. It follows that $\eta_c^R(M,N)=0$ if $[N]$ maps to a torsion element of $\underline{G}(R)$. Therefore it behooves us to find situations where $\underline{G}(R)$ is torsion.

6.2. Two-dimensional normal domains. Let $(S,\mathfrak{n},l)$ be a complete, normal domain of dimension two, and assume either

(1) l is the algebraic closure of a finite field, or
(2) l is algebraically closed of characteristic zero, and S is a rational singularity.

Then the abelian group $\underline{G}(S)$ is torsion, by [10, Proposition 2.5 and Remark 2.6].

Corollary 6.3. *Let $(R,\mathfrak{m},k)$ be a two-dimensional, excellent, normal domain containing a field, and assume that k is contained in the algebraic closure of a finite field. Assume further that R is a complete intersection of codimension $c \in \{1,2\}$.*

If M and N are finitely generated R-modules satisfying (SP_c), then they are Tor-independent.

Proof. The completion $\widehat{R}$ is an isolated singularity because R is excellent (see [27, Proposition 10.9]), and therefore $\widehat{R}$ is a normal domain. Replacing R by $\widehat{R}$, we may assume that $R = S/(\underline{f})$, where $(S, \mathfrak{n}, k)$ is a regular local ring and $\underline{f}$ is a regular sequence in $\mathfrak{n}^2$ of length c. Let $\overline{k}$ be an algebraic closure of k, and choose a *gonflement* $S \hookrightarrow (\overline{S}, \overline{\mathfrak{n}}, \overline{k})$ lifting the field extension $k \hookrightarrow \overline{k}$ (see [27, Chapter 10, §3]). This is a flat local homomorphism and is an inductive limit of étale extensions. Moreover, $\mathfrak{n}\overline{S} = \overline{\mathfrak{n}}$, so $\overline{S}$ is a regular local ring. By [27, Proposition 10.15], both $\overline{S}$ and $\overline{R} := \overline{S}/(\underline{f})$ are excellent, and $\overline{R}$ is an isolated singularity. Therefore $(\overline{R}, \overline{\mathfrak{m}}, \overline{k})$ is a normal domain. Finally, we pass to the completion T of $\widehat{S}$ and put $\Lambda = T/(\underline{f})$. This is still an isolated singularity, a normal domain, and a complete intersection of codimension c. Moreover, our hypotheses on M and N ascend along the flat local homomorphism $R \to \Lambda$, and $\eta_c^\Lambda(\Lambda \otimes_R M, \Lambda \otimes_R N) = 0$, since $\underline{G}(\Lambda)$ is torsion. By Theorem 5.11, $\operatorname{Tor}_i^\Lambda(\Lambda \otimes_R M, \Lambda \otimes_R N) = 0$ for all $i \geq 1$. The requirement on supports is automatically satisfied, since Λ is a domain (see Remarks 5.4). Faithfully flat descent completes the proof. $\square$

6.4. Even-dimensional simple singularities. Let R be a local ring whose completion is an even-dimensional simple ("ADE") singularity in characteristic zero; see, for instance, [36, Chapter 8]. Such an R is an isolated singularity, so for any pair of finitely generated R-modules M, N the length of $\operatorname{Tor}_i^R(M, N)$ is finite for $i \gg 0$. Since R is even-dimensional, $\underline{G}(\widehat{R})$ is torsion, as can be checked from [36, 13.10], and hence $\eta_1^R(M, N) = 0$. These observations are due to Dao [14, Corollary 3.16]. Thus, from Theorem 5.8 and Remark 4.9 one gets:

Corollary 6.5. *If R is as above and M, N are nonzero finitely generated R-modules with $M \otimes_R N$ torsion-free, then M, N are torsion-free, and either $\operatorname{pd}_R M$ or $\operatorname{pd}_R N$ is finite.* $\square$

6.6. Hypersurfaces. Let R be a hypersurface and an isolated singularity. Recall that the η-pairing and Hochster's θ-pairing [19] are related: $\theta(M, N) = 2\,\eta_1(M, N)$. In [14, Conjecture 3.15], see also [15], Dao has made the following

Conjecture 6.7. If $\dim R$ is even and R contains a field, then $\eta_1^R(M, N) = 0$ for all finitely generated R-modules M, N.

Moore, Piepmeyer, Spiroff and Walker [31] have settled this conjecture in the affirmative for certain types of affine algebras. Polishchuk and Vaintrob [34, Remark 4.1.5], and Buchweitz and Van Straten [8, Main Theorem], have since given other proofs, in somewhat different contexts, of this result.

Now we move on to the case of complete intersections of higher codimension.

6.8. Affine algebras over perfect fields. Here we give a localized version of a vanishing theorem for graded rings, due to Moore, Piepmeyer, Spiroff, and Walker.

Proposition 6.9. *Let k be a perfect field and $Q = k[x_1, \dots, x_n]$ the polynomial ring with the standard grading. Let $\underline{f} = f_1, \dots, f_c$ be a Q-regular sequence of homogeneous polynomials, with $c \geq 2$. Put $A = Q/(\underline{f})$ and $R = A_\mathfrak{m}$, where $\mathfrak{m} = (x_1, \dots, x_n)$. Assume that $A_\mathfrak{p}$ is a regular local ring for each $\mathfrak{p}$ in* $\operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$.

Then $\eta_c^R(M,N) = 0$ for all finitely generated R-modules M and N. In particular, if $n \geq 2c$ and the pair M, N satisfies (SP_c), then M and N are Tor-independent.

Proof. Choose finitely generated A-modules U and V such that $U_{\mathfrak{m}} \cong M$ and $V_{\mathfrak{m}} \cong N$. For any maximal ideal $\mathfrak{n} \neq \mathfrak{m}$, the local ring $A_{\mathfrak{n}}$ is regular and hence $\operatorname{Tor}_i^A(U,V)_{\mathfrak{n}} = 0$ for $i \gg 0$. It follows that the map $\operatorname{Tor}_i^A(U,V) \to \operatorname{Tor}_i^R(M,N)$ induced by the localization maps $U \to M$ and $V \to N$ is an isomorphism for $i \gg 0$. Also, for any A-module supported at $\mathfrak{m}$, its length as an A-module is equal to its length as an R-module. In conclusion, $\eta_c^R(M,N) = \eta_c^A(U,V)$.

As k is perfect, the hypothesis on A implies that the k-algebra $A_{\mathfrak{p}}$ is smooth for each non-maximal prime $\mathfrak{p}$ in A; see [16, Corollary 16.20]. Thus, the morphism of schemes $\operatorname{Spec}(R) \setminus \{\mathfrak{m}\} \to \operatorname{Spec}(k)$ is smooth. Now [32, Corollary 4.7] yields $\eta_c^A(U,V) = 0$, and hence $\eta_c^R(M,N) = 0$. It remains to note that if $n \geq 2c$, then $\dim R \geq c$, so Theorem 5.11 applies. □

Concerning complete intersections of higher codimension, in [31] the authors make the following conjecture.

Conjecture 6.10. If R is a complete intersection of codimension $c \geq 2$ and an isolated singularity, then $\eta_c^R(M,N) = 0$ for all finitely generated R-modules M, N.

Examples due to Avramov and Jorgensen [25, Example 4.1] show that the hypothesis that R be an isolated singularity is essential.

7. The Frobenius endomorphism

Let R be a Noetherian local ring of characteristic p and $\varphi\colon R \longrightarrow R$ the Frobenius endomorphism. Recall that R is *F-finite* provided φ is a finite map, that is, R is module-finite over $\varphi(R)$. Given an R-module M and a positive integer e, we write ${}^{\varphi^e}\!M$ for the R-module obtained from M by restriction of scalars along φ^e; thus $r \cdot m = r^{p^e} m$ for $r \in R$ and $m \in M$. Observe that M is torsion-free if and only if ${}^{\varphi^e}\!M$ is torsion-free for some (equivalently, all) $e \geq 1$.

We write R^{φ^e} for the R-bimodule with $r \cdot x \cdot s = rxs^{p^e}$. Then, for any R-module N, the abelian group $R^{\varphi^e} \otimes_R N$ has a structure of an R-module with action inherited from the left R-action on R^{φ^e}.

Theorem 7.1. *Assume that R is F-finite, and a complete intersection domain. The following conditions are equivalent:*

1. *$R^{\varphi^e} \otimes_R {}^{\varphi^e}\!M$ is torsion-free for every torsion-free R-module M and each integer $e \geq 1$.*
2. *$R^{\varphi^e} \otimes_R {}^{\varphi^e}\!M$ is torsion-free for some nonzero finitely generated R-module M and some integer $e \geq 1$.*
3. *R is regular.*

Proof. All three statements are true if R is a field, so we may assume that $\dim R \geq 1$. Obviously (1) $\Longrightarrow$ (2), and the implication (3) $\Longrightarrow$ (1) holds by Kunz's theorem [26, Theorem 2.1] that the R-module R^{φ^e} is flat when R is regular.

To prove that (2) $\Longrightarrow$ (3), it is helpful to write S for the target of the endomorphisms φ^e; thus

$$\varphi^e\colon R \longrightarrow S\,.$$

As φ is a finite map, so is φ^e. Applying $S \otimes_R -$ to the exact sequence

$$0 \to \top_S M \to M \to \bot_S M \to 0\,,$$

we see that $S \otimes_R M \cong S \otimes_R \perp_S M$. Moreover, $\perp_S M \neq 0$, so we can replace M by $\perp_S M$ and assume that ${}_S M$ is itself torsion-free. Since S is finite over R, M is finitely generated also over R, so one has the pushforward

$$0 \longrightarrow M \longrightarrow R^{(m)} \longrightarrow N \longrightarrow 0\,.$$

Tensoring this with S yields an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(S,N) \longrightarrow S \otimes_R M \longrightarrow S^{(m)} \longrightarrow S \otimes_R N \longrightarrow 0\,.$$

Since $S \otimes_R M$ is torsion-free and $\mathrm{Tor}_1^R(S,N)$ is torsion (recall that R is a domain), it follows that the latter module is 0, and hence that $\mathrm{Tor}_i^R(S,N) = 0$ for all $i \geq 1$, by [5, Theorem]. This implies that $\mathrm{Tor}_i^R(S,M) = 0$ for all $i \geq 1$ as well. Another application of [5, Theorem] yields that $\mathrm{pd}_R\, M$ is finite.

It remains to apply [4, Theorem 1.1] to deduce that R is regular. □

Acknowledgments

We would like to thank Hailong Dao, Anurag Singh, and Mark Walker for discussions during the preparation of this paper.

References

[1] T. Araya and Y. Yoshino, Remarks on a depth formula, a grade inequality and a conjecture of Auslander, *Comm. Algebra* **26** (1998), 3793–3806.
[2] M. Auslander, Modules over unramified regular local rings, *Illinois J. Math.* **5** (1961), 631–647.
[3] M. Auslander and C. Buchsbaum, Homological dimension in local rings. *Trans. Amer. Math. Soc.*, **85** (1957), 390–405.
[4] L. L. Avramov, M. Hochster, S. B. Iyengar, and Y. Yao, Homological invariants of modules over contracting endomorphisms, *Math. Ann.* **353** (2012), 275–291.
[5] L. L. Avramov and C. Miller, Frobenius powers of complete intersections, *Math. Res. Lett.* **8** (2001), 225–232.
[6] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1993.
[7] N. Bourbaki, *Commutative algebra*, Chapters 8 and 9, Elements of Mathematics, Springer-Verlag, Berlin, 2006.
[8] R.-O. Buchweitz and D. Van Straten, An index theorem for modules on a hypersurface singularity, *Mosc. Math. J.* **12** (2012), 237–259.
[9] O. Celikbas, Vanishing of Tor over complete intersections, *J. Comm. Alg.* **3** (2011), 169–206.
[10] O. Celikbas and H. Dao, Asymptotic behavior of Ext functors for modules of finite complete intersection dimension, *Math. Z.* **269** (2011), 1005–1020.
[11] O. Celikbas and G. Piepmeyer, Syzygies and tensor product of modules, preprint, arXiv:1210.4767.
[12] P. Constapel, Vanishing of Tor and torsion in tensor products, *Comm. Algebra* **24** (1996), 833–846.
[13] H. Dao, Asymptotic behaviour of Tor over complete intersections and applications, preprint, arxiv:07105818.
[14] H. Dao, Decent intersection and Tor-rigidity for modules over local hypersurfaces, *Trans. Amer. Math. Soc.* (to appear).
[15] H. Dao, Some observations on local and projective hypersurfaces, *Math. Res. Lett.* **15** (2008), 207–219.
[16] D. Eisenbud, *Commutative Algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 1995.
[17] R. C. Heitmann, Characterization of completions of unique factorization domains, *Trans. Amer. Math. Soc.* **337** (1993), 379–387.
[18] R. C. Heitmann and D. A. Jorgensen, Are complete intersections complete intersections? *J. Algebra* **371** (2012), 276–299.

[19] M. Hochster, The dimension of an intersection in an ambient hypersurface, In *Algebraic geometry (Chicago, Ill., 1980)*, Lecture Notes in Math. **862**, 93–106. Springer, Berlin, 1981.
[20] C. Huneke, D. A. Jorgensen, and R. Wiegand, Vanishing theorems for complete intersections, *J. Algebra*, **238** (2001), 684–702.
[21] C. Huneke and R. Wiegand, Tensor products of modules and the rigidity of Tor, *Math. Ann.* **299** (1994), 449–476; `Correction:` *Math. Ann.* **338** (2007), 291–293.
[22] C. Huneke and R. Wiegand, Correction to: "Tensor products of modules and the rigidity of Tor" [Math. Ann. **299** (1994), 449–476], *Math. Ann.*, (2007), 291–293.
[23] C. Huneke and R. Wiegand, Tensor products of modules, rigidity and local cohomology, *Math. Scand.* **81** (1997), 161–183.
[24] S. Iyengar, Depth for complexes, and intersection theorems, *Math. Z.* **230** (1999), 545–567.
[25] D. A. Jorgensen, Complexity and Tor on a complete intersection, *J. Algebra* **211** (1999), 578–598.
[26] E. Kunz, Characterization of regular local rings of characteristic p, *Amer. J. Math.* **91** (1969), 772–784.
[27] G. Leuschke and R. Wiegand, *Cohen-Macaulay Representations*, Mathematical Surveys and Monographs **181**, Amer. Math. Soc., Providence, RI 2012.
[28] S. Lichtenbaum, On the vanishing of Tor in regular local rings, *Illinois J. Math.* **10** (1966), 220–226.
[29] S. Mac Lane *Homology*, Reprint of the 1975 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
[30] C. Miller, Complexity of tensor products of modules and a theorem of Huneke-Wiegand, Proc. Amer. Math. Soc. **126** (1998), 53–60.
[31] W. F. Moore, G. Piepmeyer, S. Spiroff, and M.E. Walker, Hochster's theta invariant and the Hodge-Riemann bilinear relations, *Adv. Math.* **226** (2011), 1692–1714.
[32] W. F. Moore, G. Piepmeyer, S. Spiroff, and M.E. Walker, The vanishing of a higher codimension analog of Hochster's theta invariant, *Math. Z.*, to appear, arXiv:1110.2442.
[33] M. P. Murthy, Modules over regular local rings, *Illinois J. Math.*, **7** (1963), 558–565.
[34] A. Polishchuk and A. Vaintrob, Chern characters and Hirzebruch-Riemann-Roch formula for matrix factorizations, *Duke Math. J.* **161** (2012), 1863–1926.
[35] W. V. Vasconcelos, Reflexive modules over Gorenstein rings, *Proc. Amer. Math. Soc.*, **19** (1968), 1349–1355.
[36] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series **146**, Cambridge University Press, Cambridge, 1990.

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